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Compressible Alfven turbulence in one dimension

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We present the simplest extension of the Burgers equation to include the effects of magnetic pressure. For unity magnetic Prandtl number, an exact solution exists that describes Alfvenic shock waves. For forced turbulence with arbitrary diffusivities, renormalized perturbation theory is used to show that the only stable, physically accessible fixed point corresponds to a state of equidissipation. Energy equipartition, however, requires the equality of the forcing functions. Implications for the spectra of turbulence and self-organization phenomena in magnetohydrodynamics are discussed. [S1063-651X(98)50109-2]

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A complete understanding of spatiotemporal intermittency remains the most elusive problem in turbulence theory. Intermittency phenomena, which usually involve coherent structures in space and time, complicate the conventional wisdom based on scaling arguments, etc. Indeed, intermittency phenomena are *characterized* by coherency over ranges of scales (large or small) and higher order (rather than quadratic) correlations, manifestly at variance with the K41 assumptions of homogeneity, scale similarity, and statistical isotropy. Many attempts to explain these deviations involve geometric corrections to the energy spectrum but lose touch with the underlying dynamical system. Another route to insight into the development of structure may be gained from simplified models, such as the Burgers equation [1], which allows a deeper investigation of the relevant fundamental dynamics. While both dynamical and geometrical approaches have been extensively applied to intermittent hydrodynamic turbulence [2–5], only the mean-field scaling approach has been used for magnetohydrodynamics (MHD) turbulence, until now [6–8]. Indeed, the field of intermittency phenomena in MHD turbulence is largely uncharted territory, despite its relevance to astrophysics, geophysics, and technological applications (e.g., controlled fusion, drag reduction, etc.). This is unfortunate from a theoretical viewpoint as well, since the addition of a magnetic field can increase the possible pathways to structure formation.

To address these issues, a simplified, analytically tractable model of MHD turbulence is needed. In this spirit, we propose an extension of the Burgers model of compressible fluid turbulence, or ''Burgerlence,'' to include the effects of magnetic pressure. The fluid dynamics (in the $\hat{x}$ direction, for example) are governed by the momentum equation

$$\frac{\partial v}{\partial t}+\lambda_v v\,\frac{\partial v}{\partial x}=-\lambda_B\,\frac{\partial}{\partial x}\left(\frac{B^2}{8\pi\rho_0}\right)+\nu\,\frac{\partial^2 v}{\partial x^2}+\tilde{f}_v, \qquad (1)$$

where ν is the kinematic viscosity and $\tilde{f}_v$ is a (optional) random forcing function. λ_v and λ_B are theoretical dials introduced to measure the strengths of the nonlinearities (they will be set to one at the end of the calculations). It is assumed that the magnetic pressure $(B^2)/8\pi$ is much greater than the fluid pressure, and that the density variations are negligible compared with those of the magnetic field (this is discussed presently). The magnetic field itself evolves according to the induction-diffusion equation, which in one dimension reduces to

$$\frac{\partial B}{\partial t}=-\lambda_B\,\frac{\partial(vB)}{\partial x}+\eta\,\frac{\partial^2 B}{\partial x^2}+\tilde{f}_B. \qquad (2)$$

Here, $\eta=c^2/4\pi\sigma$ is the magnetic diffusivity, σ is the electrical conductivity, λ_B is the Lorentz force coupling strength, and $\tilde{f}_B$ represents random ''seeding'' of the magnetic field.

In the absence of dissipation and forcing, Eqs. (1) and (2) conserve energy (kinetic plus magnetic) and the magnetic flux $\int vB\,dx$. Hence, the model is the simplest possible set of equations that allows Alfvenization, i.e., the interchange of magnetic and fluid energies, consistent with energy conservation. The inclusion of compressional evolution of the fluid density only complicates this simple, basic picture, motivating us to assume a constant background density throughout. By analogy with the Burgers model of compressible turbulence, Eqs. (1) and (2) are referred to as the MHD Burgerlence model.

Without active sources, Eqs. (1) and (2) represent the ''decay'' problem of arbitrary initial conditions. For the special case of equal dissipation, $\nu=\eta$, the system reduces to two decoupled Burgers equations in the Elsasser variables $v\pm B$. This is not surprising, since in dissipationless MHD Burgerlence, initial value data is propagated along the characteristics $dx/dt=v\pm v_A$ at the constant characteristic velocities $v\pm v_A$, where $v_A=\sqrt{B/(4\pi n_0 m)}$. All of the familiar results from the Burgers equation may be applied to this special case. In particular, the well-known exact solutions imply that the system can support Alfvenized shock waves. Note that in hydrodynamic Burgers turbulence, regions of negative velocity gradient steepen into shock singularities, while $\acute{v}>0$ regions smooth out. This asymmetry in $\acute{v}$ evolution is the origin of the asymmetric probability distribution function (PDF) of $\acute{v}$ observed in Burgerlence [9–12]. In the MHD case, conservation of magnetic flux requires a concentration of magnetic field at the velocity shock fronts (limiting fluid wave steepening through pressure back-reaction). Moreover, the relation $\partial_t(B_x)\approx-2(v_x B_x)$ implies that both negative and *positive* magnetic shocks are possible. Since these shocks dominate the energy spectrum of the system [13], magnetic intensity in MHD Burgerlence is inherently intermittent.

It can be shown that the more general case $\eta \neq \nu$ is not integrable. Thus, we proceed directly to the case of noisy Burgerlence, where random forcing drives the above shock production. The presence of forcing highlights several dynamic regimes.

(1) $\tilde{f}_v \neq 0$, $\tilde{f}_B = 0$: the fluid is actively stirred, while B is convected. For low magnetic fields, pressure back-reaction is negligible, and the system reduces to Burgers advection of a passive scalar.

(2) $\tilde{f}_v = 0$, $\tilde{f}_B \neq 0$: the magnetic field has an active source and the fluid responds to the induced pressure. Obviously, this is a B^2 (i.e., higher-order) effect.

(3) $\tilde{f}_v \neq 0$, $\tilde{f}_B \neq 0$: fully driven turbulence.

The ''typical'' MHD approach is case (1), in which fluid forcing at large scales produces a Kolmogorov-type energy cascade. In Burgers turbulence, small-scale disturbances directly affect large-scale structures (through shocks), so forcing at all scales is the standard statistical tool. Here we actively excite both fields, treating v and $B = B_\perp$ as fluctuations above a uniform, force-free ($v \| B_0$) configuration.

This broadband forcing, externally imposed or internally self-generated, replaces the ''inertial'' range with a range of dynamic turbulence. This view allows the physically interesting question, ''Given a turbulent energy spectrum, what type of forcing will reproduce it?'' The answer is intimately related to the dynamics of the system, since the nonlinearities (common to both the Burgers and higher-dimensional models) will distort the symmetries and statistics of the source. Indeed, these nonlinear couplings will generate asymmetric, non-Gaussian PDF's, even for a white-noise forcing spectrum. These deviations from normality (e.g., shock formation) are the hallmarks of intermittency. Then, to simplify the analysis, we first assume Gaussian noise spectra, with $\langle \tilde{f}^2_{v,B} \rangle = S_{v,B}$ and $\langle \tilde{f}_v \tilde{f}_B \rangle = 0$, i.e., no cross correlation. The extension to spatially dependent noise will follow.

We are interested in MHD Burgerlence for long times and large distances. For homogeneous turbulence in the inertial range, there are no intrinsic scale lengths. Dynamical terms will dominate beyond the dissipative lengths, and correlation functions asymptotically approach simple algebraic forms [14,15]. For example, the velocity autocorrelation $\langle \delta v^2(\delta x, t) \rangle$ has the homogeneous form $(\delta x)^{-\alpha} \langle \delta v^2(t/\delta x^a) \rangle$. Alternatively, $\omega \propto k^a$ may be viewed as a nonlinear dispersion relation for the system [16]. To test the dependence of the various parameters on scale, assume that we change the length scale $x \to bx$. With this similarity transformation, the other variables scale as $t \to b^a t$, $v \to b^c v$, $B \to b^d B$. The parameters of Eqs. (1) and (2) thus become

$$\begin{Bmatrix} \lambda_v \\ \lambda_B \end{Bmatrix} \to b^{[3(a-1)]/2} \begin{Bmatrix} \lambda_v \\ \lambda_B \end{Bmatrix}, \quad \begin{Bmatrix} \nu \\ \eta \end{Bmatrix} \to b^{a-2} \begin{Bmatrix} \nu \\ \eta \end{Bmatrix}. \tag{3}$$

Here we have noted that consistent scaling of λ_B implies that $c = d$. Therefore, v and B scale the same way (necessary for the conservation of energy). Moreover, the assumption of white noise implies that $\langle \tilde{f}^2 \rangle = \int \tilde{f}^2 dk\, d\omega$ is invariant to a change in scale. Hence, $a = 2c + 1$ and there is only one independent exponent to find. However, there is one more scaling constraint: invariance under the Galilean transformation $v \to v(x - ut, t) + u$, $B \to B(x - ut, t)$. This symmetry is well known for ideal MHD, where the ''frozen-in'' law assures us that the magnetic field transforms identically with the fluid [17]. Mathematically, however, the invariance arises from cancellation between the nonlinearities and the time derivatives, a balance that is not upset by the addition of dissipative terms. This cancellation is crucially important to perturbative schemes based on the nonlinear interactions, since *Galilean invariance precludes renormalization of the coupling coefficients* [18]. This constraint immediately leads to the scaling exponents $a = 1, c = 0$. That is, $x \sim t$, so the transport is ballistic rather than diffusive, as in the case of hydrodynamic Burgerlence. The *speed* of propagation, though, can only be determined through approximation methods.

To explore the dynamics, we employ renormalized perturbation theory:

$$v_{k,\omega} = v^{(0)}_{k,\omega} + \lambda_v v^{(1)}_{k,\omega} + \lambda_v^2 v^{(2)}_{k,\omega} + \cdots ,$$
$$B_{k,\omega} = B^{(0)}_{k,\omega} + \lambda_B B^{(1)}_{k,\omega} + \lambda_B^2 B^{(2)}_{k,\omega} + \cdots . \tag{4}$$

Using standard techniques [14,18], the perturbation effects on the Green's functions may be absorbed into effective transport coefficients (ν and η). To second order in $\lambda_{v,B}$, the renormalized viscosity and resistivity (as $k, \omega \to 0$) are

$$\nu^t = \frac{1}{4\pi^2} \int dk'\, d\omega' [\lambda_v^2 G_0^v(k',\omega') |v^{(0)}_{k',\omega'}|^2 + \lambda_B^2 G_0^B(k',\omega') |B^{(0)}_{k',\omega'}|^2] \to \frac{1}{4\pi} \left[\frac{\lambda_v S_v}{\nu^2} + \frac{\lambda_B S_B}{\eta^2} \right] \int_{k_{\min}}^{\infty} \frac{dk'}{k'^4}, \tag{5}$$

$$\eta^t = \frac{\lambda_B^2}{8\pi^2} \int dk'\, d\omega' [G_0^B(k',\omega') |v^{(0)}_{k',\omega'}|^2 + G_0^v(k',\omega') |B^{(0)}_{k',\omega'}|^2] \to \frac{\lambda_B^2}{2\pi(\eta+\nu)} \left[\frac{S_v}{\nu} + \frac{S_B}{\eta} \right] \int_{k_{\min}}^{\infty} \frac{dk'}{k'^4}. \tag{6}$$

Here, $v^{(0)}_{k,\omega} = G_0^v(k,\omega) \tilde{f}_v \equiv [1/(-i\omega + \nu k^2)] \tilde{f}_v$ and $B^{(0)}_{k,\omega} = G_0^B(k,\omega) \tilde{f}_B \equiv [1/(-i\omega + \eta k^2)] \tilde{f}_B$ define the bare (unrenormalized) propagators G_0^v and G_0^B, and S_v and S_B are the (white) noise strengths of the forcing functions. An infrared cutoff $k_{\min}$ has been introduced to prevent the divergence of slow modes.

In the inertial range, these turbulent diffusivities dominate the original bare ones. Letting $\nu \to \nu^t$ and $\eta \to \eta^t$, Eqs. (5) and (6) become self-consistent recursion relations for the effective viscosity and diffusivity. In terms of the dimensionless interaction parameters $U_1 = (\lambda_v^2 S_v)/[6\pi k^3_{\min} (\nu^t)^3]$ and $U_2 = (\lambda_B^2 S_B)/[6\pi k^3_{\min} (\eta^t)^3]$, the fixed points are

$$(U_1,U_2)_1=\left(1-\sqrt{1-r},1-\frac{2}{r}\,(1+\sqrt{1-r})\right), \quad (7a)$$

$$(U_1,U_2)_2=\left(1+\sqrt{1-r},1-\frac{2}{r}\,(1-\sqrt{1-r})\right), \quad (7b)$$

$$(U_1,U_2)_3=\left(\frac{2}{1+r},\frac{2r}{1+r}\right). \quad (7c)$$

Here, the ratio of the noise strengths $r\equiv S_B/S_v$ is the only independent parameter. Note that $0\leqslant r\leqslant\infty$. In particular, r may be greater than one, implying that the first two solutions may give complex diffusivities. Imaginary components in the transport coefficients suggest nonlinear frequency shifts, i.e., the propagation of Alfven waves, so solutions (7a) and (7b) cannot be ruled out as unphysical, *a priori*. Solution (7c) gives strictly dissipative behavior. A simple calculation shows that solution (7c) is linearly stable for all r, while (7a) and (7b) are only stable for $r\geqslant 5.3$. The question then becomes one of physical accessibility. In other words, given a set of meaningful initial conditions, which asymptotic fixed point will the system approach? To determine this convergence, we need an analysis of the phase flow in solution space, i.e., a set of evolution equations for the effective diffusivities.

The dynamical renormalization group yields such a phase space description by successively summing the modal interactions over bands of spatial scales. Specifically, the integrations in Eqs. (5) and (6) are performed over a shell of momenta $k_{\min}e^{-\delta l}\approx k_{\min}(1-\delta l)\leqslant k\leqslant k_{\min}$, where δl is infinitesimal. The system is then rescaled as $k\to ke^{-\delta l}$. This is the same scaling as before, with $b=e^{\delta l}$. This transformation establishes differential recursion relations for the transport coefficients, which give (to first order in δl)

$$\frac{dU_1}{dl}=3U_1\left[1-\frac{U_1}{2}-\left(\frac{rU_1}{U_2}\right)^{1/3}U_2\right], \quad (8)$$

$$\frac{dU_2}{dl}=3U_2\left[1-\frac{U_1(U_2/rU_1)^{1/3}+U_2(rU_1/U_2)^{1/3}}{1+(rU_1/U_2)^{1/3}}\right]. \quad (9)$$

The fixed points of these equations are given by the solutions (7). There are two ranges to consider: (1) $r>1$, giving one real and two complex conjugate solutions, and (2) $r\leqslant 1$, giving three real solutions. Since the recursion relations (8) and (9) are both real, no real initial parameters (U_1,U_2) can evolve to a complex fixed point. Then, in the first regime only solution (7c) is physically accessible.

For $r\leqslant 1$, there is one positive solution and two negative ones for U_2. Figure 1 shows the first quadrant of a phase flow diagram for the representative value $r=\frac{1}{2}$. The arrows indicate the flow under the renormalization transformations (8) and (9). Note, in particular, that the axes are repellors. Thus, for any physical starting point $(\nu,\eta)>0$, only the positive fixed point is accessible. Once again, solution (7c) appears as the unique infrared fixed point for the system.

Using these results, the turbulent transport coefficients are given by

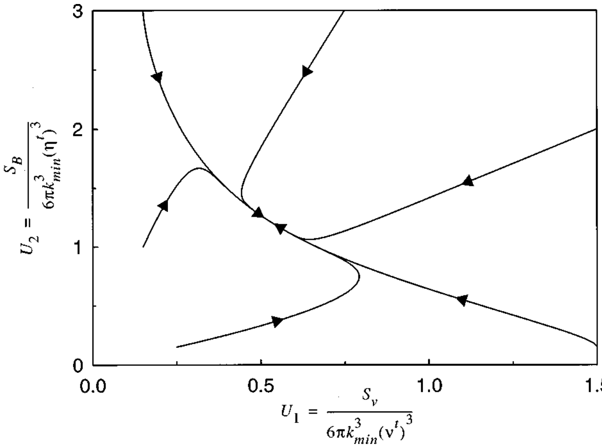

FIG. 1. Renormalization phase flow diagram of the dimensionless interaction parameters $U_1=S_v/[6\pi k_{\min}^3(\nu^t)^3]$ and $U_2=S_B/[6\pi k_{\min}^3(\eta^t)^3]$ for the representative value $r\equiv S_B/S_v=\frac{1}{2}$. The trajectories are defined by Eqs. (8) and (9).

$$\nu^t=\eta^t=\left[\frac{S_v+S_B}{12\pi}\right]^{1/3}k_{\min}^{-1}. \quad (10)$$

Then, in the inertial range the turbulent fluid viscosity and magnetic diffusivity are equal. In other words, as the fluid transport rate increases due to nonlinear interactions, the magnetic field is convected along faster as well. Of course, the enhanced magnetic diffusivity backreacts on the fluid, dragging it along at a faster rate. The net result is a balance between the two effective diffusivities.

However, equality of the turbulent dissipation does not imply the equipartition of fluid and magnetic energy, as is commonly assumed. A straightforward calculation shows that the energy spectra are given by

$$E_v(k)\equiv\frac{1}{2}\,\rho_0\langle\tilde{v}^2\rangle_k=\frac{1}{2}\,\rho_0\left[\frac{3\pi}{2(S_v+S_B)}\right]^{1/3}S_vk^{-1}+C_2S_B^{2/3}k^{-1}, \quad (11)$$

$$E_B(k)\equiv\frac{1}{2}\,\rho_0\langle\tilde{B}^2\rangle_k=\frac{1}{2}\,\rho_0\left[\frac{3\pi}{2(S_v+S_B)}\right]^{1/3}S_Bk^{-1}, \quad (12)$$

where $C_2=[(9/2)(1-ln2)+5\pi/\sqrt{3}](12\pi)^{1/3}$ is important only when $S_B\gg S_v$, i.e., when magnetic forcing dominates the fluid motion through pressure effects. For the more standard case of significant fluid forcing, the equipartition of energy only occurs if $S_v=S_B$ (a conclusion that also holds for spatially dependent noise). This distinction between equal dissipation and energy equipartition has been observed in three-dimensional simulations of incompressible MHD as well [19].

Note that the rather weak spatial falloff of these spectra indicate the significance of small-scale noise on large scales. In particular, the forcing of small scales present in the white-noise spectra inhibits shock formation, reducing the wave-number dependence from k^{-2} to k^{-1}. For the more general case of spatially dependent noise, only power-law singularities $S_{v,B}\sim k^{-2\beta}$ are relevant in the asymptotic $(k,\omega\to 0)$ limit. The resulting energy spectra scale as $E_{v,B}(k)\sim k^{-1-(4\beta/3)}$, a relation that has been verified numerically [20]. Appropriate values of β can give Kolmogorov or

Kraichnan-Iroshnikov (KI) (or any other) scaling. This latter reproduction is particularly interesting, since the KI theory emphasizes the effect of a large-scale field on small-scale energy transfer (the opposite limit is considered here). This distinction is fortunate from the viewpoint of self-organization phenomena (e.g., magnetic dynamos, shear-induced mean flow, etc.), since energy transfer from fluid to field at large scales seems unlikely given the constraint of equipartition. An alternative scenario for a large-scale structure is amplification by equidissipation turbulence, followed by the nonlinear saturation of growth. In time, the saturated state might then relax towards equipartition of energy.

It would be interesting to see if this equidissipation state is extended beyond the ''inertial'' range. That is, for nontrivial initial diffusivities, will the system dynamically self-adjust to maintain $\nu+\nu^t=\eta+\eta^t$? This would place a fundamental constraint on the *onset* of intermittency as well. A related concern is the probability distribution of v and B for the general case $\nu\neq\eta$. In fully developed MHD Burgerlence, the equidissipation state leads to PDF asymmetry in the characteristic variables $z_\pm=v\pm v_A$. Before saturation, however, the governing statistics remain an open question. For the generation and maintenance of self-organized structures, it is these PDF's that are needed most. The determination of these distributions and a classification of their associated structures will be the subject of future papers.

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